Laplace transformations of hydrodynamic-type systems in Riemann invariants: Periodic sequences

1997 J. Phys. A: Math. Gen. 30 6861

(http://iopscience.iop.org/0305-4470/30/19/023)

J. Phys. A: Math. Gen. **30** (1997) 6861–6878. Printed in the UK PII: S0305-4470(97)84392-2

Laplace transformations of hydrodynamic-type systems in Riemann invariants: Periodic sequences

E V Ferapontov†‡
Institute for Mathematical Modelling, Academy of Science of Russia, Miusskaya, 4, 125047 Moscow, Russia

Received 20 May 1997

Abstract. The conserved densities of hydrodynamic-type systems in Riemann invariants satisfy a system of linear second-order partial differential equations. For linear systems of this type Darboux introduced Laplace transformations, generalizing the classical transformations in the scalar case. It is demonstrated that Laplace transformations can be pulled back to the transformations of the corresponding hydrodynamic-type systems. We discuss periodic Laplace sequences of 2×2 hydrodynamic-type systems with emphasis on the simplest nontrivial case of period 2.

For 3×3 systems in Riemann invariants a complete discription of closed quadruples is proposed. They turn out to be related to a special quadratic reduction of the $(2+1)$-dimensional 3-wave system which can be reduced to a triple of pairwise commuting Monge–Ampere equations.

In terms of the Lame and rotation coefficients Laplace transformations have a natural interpretation as the symmetries of the Dirac operator, associated with the $(2+1)$-dimensional n-wave system. The 2-component Laplace transformations can also be interpreted as the symmetries of the $(2+1)$-dimensional integrable equations of Davey–Stewartson type.

Laplace transformations of hydrodynamic-type systems originate from a canonical geometric correspondence between systems of conservation laws and line congruences in projective space.

1. Introduction and the main results

Let us consider a 2-component system of hydrodynamic type in Riemann invariants

$$\begin{aligned} R^1_t &= \lambda^1(R) R^1_x \\ R^2_t &= \lambda^2(R) R^2_x. \end{aligned} \tag{1}$$

Any system (1) possesses infinitely many conservation laws of hydrodynamic type $\int u(R)\,\mathrm{d}x$ with the conserved densities $u(R)$ satisfying linear equation of the second order

$$\partial_1 \partial_2 u = a \partial_1 u + b \partial_2 u \tag{2}$$

where

$$a = \frac{\partial_2 \lambda^1}{\lambda^2 - \lambda^1} \qquad b = \frac{\partial_1 \lambda^2}{\lambda^1 - \lambda^2}$$

$\partial_i = \frac{\partial}{\partial R^i}$. Let also f be the flux, corresponding to the density u, that is, f satisfies the equation $u_t = f_x$ or, equivalently,

$$\partial_i f = \lambda^i \partial_i u \qquad \text{for any } i = 1, 2$$

† E-mail address: fer@landau.ac.ru
‡ Present address: Fachbereich Mathematik, SFB 288, Technische Universität Berlin, 10623 Berlin, Germany. E-mail address: fer@sfb288.math.tu-berlin.de

which are compatible due to (2).

Applying to equation (2) the Laplace transformation

$$U = u - \frac{\partial_1 u}{b}$$

we arrive at the similar equation with respect to U:

$$\partial_1 \partial_2 U = A \partial_1 U + B \partial_2 U$$

where

$$A = a - \partial_2 \ln b \qquad B = b + \partial_1 \ln A. \tag{3}$$

(the inverse Laplace transformation $U = u - \frac{\partial_2 u}{a}$ is considered analogously). It turns out that Laplace transformations can be pulled back to the transformations of the corresponding hydrodynamic-type systems. Let us introduce the system

$$\begin{aligned} R^1_t &= \Lambda^1(R) R^1_x \\ R^2_t &= \Lambda^2(R) R^2_x \end{aligned} \tag{4}$$

where the new characteristic velocities Λ^1, Λ^2 are connected with λ^1, λ^2 by the formulae

$$\begin{aligned} \Lambda^1 &= \lambda^2 \\ \Lambda^2 &= \lambda^2 - \frac{b \partial_2 \lambda^2}{\partial_2 b - ab} = \lambda^2 - \frac{1}{\frac{\partial_1 \partial_2 \lambda^2}{\partial_1 \lambda^2 \partial_2 \lambda^2} + \frac{1}{\lambda^1 - \lambda^2}}. \end{aligned} \tag{5}$$

Theorem 1. (1) The conserved densities U of system (4) are given by the formula

$$U = u - \frac{\partial_1 u}{b}$$

where u are conserved densities of system (1).

(2) The characteristic velocities W^1, W^2 of commuting flows of system (4) are given by the formulae

$$\begin{aligned} W^1 &= w^2 \\ W^2 &= w^2 - \frac{b \partial_2 w^2}{\partial_2 b - ab} \end{aligned}$$

where w^1 and w^2 are the characteristic velocities of commuting flows of system (1), i.e. solutions of the linear system (see [1])

$$\frac{\partial_2 w^1}{w^2 - w^1} = a \qquad \frac{\partial_1 w^2}{w^1 - w^2} = b.$$

(3) The flux F of the conserved density U is given by the formula

$$F = f - \lambda^2 \frac{\partial_1 u}{b}$$

where f is the flux of u.

The proof follows from the identities

$$\begin{aligned} A &= \frac{\partial_2 \Lambda^1}{\Lambda^2 - \Lambda^1} \qquad & B &= \frac{\partial_1 \Lambda^2}{\Lambda^1 - \Lambda^2} \\ \frac{\partial_2 W^1}{W^2 - W^1} &= A \qquad & \frac{\partial_1 W^2}{W^1 - W^2} &= B \end{aligned}$$

and

$$\partial_i F = \Lambda^i \partial_i U \qquad i = 1, 2$$

which can be checked by a direct calculation. It is natural to call system (4) the Laplace transformation of system (1). Evidently Laplace transformations preserve the 'integrability': the conserved densities and commuting flows of system (1) are automatically transformed into the conserved densities and commuting flows of system (4) according to the formulae of theorem 1. In particular, solutions of system (1), specified by the hodograph formula (see [1])

$$w^1 = x + \lambda^1 t \qquad w^2 = x + \lambda^2 t$$

are transformed into the solutions

$$W^1 = x + \Lambda^1 t \qquad W^2 = x + \Lambda^2 t$$

of system (4).

The inverse Laplace transformation $U = u - \frac{\partial_2 u}{a}$ corresponds to the interchange of indices 1 and 2 in formulae (5). Some further properties of Laplace transformations in the 2-component case are discussed in section 2.

Following [4] in section 5 we give the interpretation of Laplace transformation (5) as a discrete symmetry of the integrable equations of Davey–Stewartson type.

Formula (5) results from the geometric construction of paper [16], relating systems of conservation laws with line congruences in the projective space. This correspondence is briefly discussed in section 6.

Laplace transformations can be generalized to $n \times n$ systems in Riemann invariants

$$R^i_t = \lambda^i(R) R^i_x \tag{6}$$

namely, for any pair of indices $i \neq j$ we define transformation S_{ij}, mapping system (6) into the new system

$$R^i_t = \Lambda^i(R) R^i_x \tag{7}$$

with the characteristic velocities

$$\begin{aligned}
\Lambda^i &= \lambda^j \\
\Lambda^j &= \lambda^j - \frac{a_{ji}\partial_j \lambda^j}{\partial_j a_{ji} - a_{ij}a_{ji}} \\
\Lambda^k &= \frac{\lambda^k a_{ji} - \lambda^j a_{ki}}{a_{ji} - a_{ki}} \qquad k \neq i, j
\end{aligned} \tag{8}$$

where $a_{ji} = \frac{\partial_i \lambda^j}{\lambda^i - \lambda^j}$.

Theorem 2. (1) The conserved densities U of system (7) are given by the formula

$$U = u - \frac{\partial_i u}{a_{ji}}$$

where u are conserved densities of system (6).

(2) The characteristic velocities W^i of commuting flows of system (7) are given by the formulae

$$\begin{aligned}
W^i &= w^j \\
W^j &= w^j - \frac{a_{ji}\partial_j w^j}{\partial_j a_{ji} - a_{ij}a_{ji}} \\
W^k &= \frac{w^k a_{ji} - w^j a_{ki}}{a_{ji} - a_{ki}} \qquad k \neq i, j
\end{aligned}$$

where w^i are the characreristic velocities of commuting flows of system (6), i.e. solutions of the linear system (see [1])

$$\frac{\partial_j w^i}{w^j - w^i} = a_{ij} \qquad i \neq j.$$

(3) The flux F of the conserved density U of system (7) is given by the formula

$$F = f - \lambda^j \frac{\partial_i u}{a_{ji}}$$

where f is the flux of u.

We recall that the conserved densities u of system (6) satisfy an over-determined system of linear second-order equations

$$\partial_i \partial_j u = a_{ij} \partial_i u + a_{ji} \partial_j u \qquad i \neq j \tag{9}$$

with the compatibility conditions

$$\partial_k a_{ij} = a_{ik} a_{kj} + a_{ij} a_{jk} - a_{ij} a_{ik} \qquad i \neq j \neq k \neq i$$

which we always assume to be satisfied. Systems (6), satisfying these compatibility conditions, are called semi-Hamiltonian and can be integrated by the generalized hodograph transform [1]. The formula

$$U = u - \frac{\partial_i u}{a_{ji}}$$

defines Laplace transformation S_{ij} of linear system (9)—see Darboux [5, p 274]. As one can verify directly, U satisfies the system

$$\partial_i \partial_j U = A_{ij} \partial_i U + A_{ji} \partial_j U \qquad i \neq j$$

with the coefficients A given by the formulae

$$\begin{aligned}
A_{ij} &= a_{ij} - \partial_j \ln a_{ji} \\
A_{ji} &= a_{ji} + \partial_i \ln A_{ij} \\
A_{ik} &= a_{jk} \left(1 - \frac{a_{ki}}{a_{ji}}\right) \\
A_{ki} &= a_{ki} + \partial_i \ln \left(1 - \frac{a_{ki}}{a_{ji}}\right) \\
A_{jk} &= a_{jk} + \partial_k \ln A_{ij} \\
A_{kj} &= a_{kj} + \partial_j \ln \left(1 - \frac{a_{ki}}{a_{ji}}\right) \\
A_{kl} &= a_{kl} + \partial_l \ln \left(1 - \frac{a_{ki}}{a_{ji}}\right)
\end{aligned} \tag{10}$$

where $k, l \neq i, j$ (compare with [13, 14]). The nonsymmetry of these expressions is due to the distinguished role played by the indices i and j in the definition of the Laplace transformation S_{ij}. Laplace transformations preserve the semi-Hamiltonian property and hence map integrable systems to integrable.

In the case $n = 2$ there is actually only one transformation S_{12} since $S_{12} \circ S_{21} = \mathrm{id}$.

Some further properties of transformations S_{ij} are discussed in section 3, where we also propose a complete description of quadruples of 3×3 systems, which are closed under all Laplace transformations S_{ij}.

In section 4 formulae for Laplace transformations of the Lame and rotation coefficients are presented.

2. Laplace transformations of 2-component systems. Sequences of period 2

Iterating Laplace transformations according to formulae (5) we arrive at the infinite sequence of systems with the characteristic velocities being consecutive solutions of the integrable chain

$$\frac{\partial_1\partial_2\lambda^n}{\partial_1\lambda^n\partial_2\lambda^n}=\frac{1}{\lambda^n-\lambda^{n-1}}+\frac{1}{\lambda^n-\lambda^{n+1}} \tag{11}$$

so that the Laplace transformation of the system with characteristic velocities $(\lambda^{n-1},\lambda^n)$ is the system with characteristic velocities $(\lambda^n,\lambda^{n+1})$. In terms of the coefficients

$$a^n=\frac{\partial_2\lambda^n}{\lambda^{n+1}-\lambda^n}\qquad b^n=\frac{\partial_1\lambda^n}{\lambda^{n-1}-\lambda^n}$$

chain (11) assumes the form

$$\partial_1 a^n=a^n(b^{n+1}-b^n)\qquad \partial_2 b^n=b^n(a^{n-1}-a^n)$$

and after substitution

$$q^n=\ln a^nb^n=\ln\frac{\partial_2\lambda^n\partial_1\lambda^n}{(\lambda^{n+1}-\lambda^n)(\lambda^{n-1}-\lambda^n)}$$

reduces to the well known Toda chain

$$\partial_1\partial_2 q^n=2\mathrm{e}^{q^n}-\mathrm{e}^{q^{n+1}}-\mathrm{e}^{q^{n-1}}. \tag{12}$$

Chain (11) recently appeared in [4] as a symmetry of the $(2+1)$-dimensional integrable equations of Davey–Stewartson type—see section 5.

A number of interesting results in the theory of Laplace transformations have been derived while studying periodic sequences. It was demonstrated in [3] that periodic sequences are ultimately connected with the spectral theory of the two-dimensional Schrödinger operator. It turns out that any periodic sequence of Laplace transformations of second-order equations (2) can be pulled back to the periodic sequence of Laplace transformations of the corresponding hydrodynamic-type systems. This pull-back is governed by periodic reductions of chain (11).

Example. Let us consider a periodic sequence of Laplace transformations of period 2, where equation (2) with coefficients (a,b) first transforms into equation (2) with coefficients (A,B), and then back into (a,b). It follows from (3) that $Aa=\varphi_2(R^2)$, $Bb=\varphi_1(R^1)$, where the functions φ_i can be reduced to ±1 by a change to the new Riemann invariants, so that we can assume $Aa=\pm1$, $Bb=\pm1$. In what follows we consider the case $A=\frac{1}{a}$, $B=\frac{1}{b}$ so that our Laplace sequence can be schematically represented as follows

$$(a,b)\to\left(\frac{1}{a},\frac{1}{b}\right)\to(a,b). \tag{13}$$

Moreover, the coefficients a and b must obey the equations

$$\partial_2\ln b=a-\frac{1}{a}\qquad \partial_1\ln a=b-\frac{1}{b} \tag{14}$$

which are equivalent to the sh-Gordon equation

$$\partial_1\partial_2\varphi=4\,\mathrm{sh}\,\varphi$$

for $\varphi=\ln ab$. The periodic pull-back, corresponding to sequence (13), is of the form

$$(\lambda^1,\lambda^2)\to(\Lambda^1,\Lambda^2)\to(\lambda^1,\lambda^2)$$

where $\Lambda^1 = \lambda^2$, $\Lambda^2 = \lambda^1$, as automatically follows from (5). Here $a, b, \lambda^1, \lambda^2$ are connected by the formulae

$$\frac{\partial_2\lambda^1}{\lambda^2-\lambda^1} = a \qquad \frac{\partial_1\lambda^2}{\lambda^1-\lambda^2} = b$$
$$\frac{\partial_2\lambda^2}{\lambda^1-\lambda^2} = \frac{1}{a} \qquad \frac{\partial_1\lambda^1}{\lambda^2-\lambda^1} = \frac{1}{b} \tag{15}$$

which are compatible due to (14). One can show that λ^1, λ^2 satisfy the second-order system

$$\frac{\partial_1\partial_2\lambda^1}{\partial_1\lambda^1\partial_2\lambda^1} = \frac{2}{\lambda^1-\lambda^2} \qquad \frac{\partial_1\partial_2\lambda^2}{\partial_1\lambda^2\partial_2\lambda^2} = \frac{2}{\lambda^2-\lambda^1}$$

which is just a periodic reduction of chain (11) of period 2: $\lambda^3 = \lambda^1$, $\lambda^4 = \lambda^2$ and can be obtained by varying the Lagrangian

$$L = \int\int \frac{\partial_1\lambda^1\partial_2\lambda^2}{(\lambda^1-\lambda^2)^2}\,\mathrm{d}R^1\,\mathrm{d}R^2.$$

Formulae (15) describe the periodic pull-back of Laplace sequence of period 2. This pull-back is defined uniquely up to transformations $\lambda^1 \to p\lambda^1 + q$, $\lambda^2 \to p\lambda^2 + q$, $p, q =$ constant. Let us point out that applying Laplace transformations to commuting flows with the characteristic velocities w^1, w^2, which do not satisfy the restrictions (15) (although correspond to the same a, b), we will not return back after going round the cycle.

Remark. For any periodic sequence of Laplace transformations of period 2

$$(a, b) \to \left(\frac{1}{a}, \frac{1}{b}\right) \to (a, b)$$

one can construct a nonperiodic pull-back of the form

$$(w^1, w^2) \to (w^2, \mu w^1) \to (\mu w^1, \mu w^2)$$

where $\mu =$ constant. Here a, b, w^1, w^2 are connected by the formulae

$$\frac{\partial_2 w^1}{w^2-w^1} = a \qquad \frac{\partial_1 w^2}{w^1-w^2} = b$$
$$\frac{\partial_2 w^2}{\mu w^1-w^2} = \frac{1}{a} \qquad \frac{\mu\partial_1 w^1}{w^2-\mu w^1} = \frac{1}{b}. \tag{16}$$

The linear system (16) is compatible and manifests the spectral problem for equations (14) with the spectral parameter μ. It follows from (16) that (λ^1, λ^2) are the components of the wavefunction (w^1, w^2) at the point $\mu = 1$.

The situation with periodic sequences of an arbitrary period n is completely analogous, namely, for any periodic sequence of Laplace transformations of second-order equations of period n there exists exactly n-parameter family of periodic sequences of systems of hydrodynamic type with the same period. In a similar way (considering nonperiodic pull-backs) one can construct spectral problems, corresponding to periodic sequences of Laplace transformations of an arbitrary period n.

Let us now discuss another question, concerning Laplace sequences of period 2

$$(a, b) \to \left(\frac{1}{a}, \frac{1}{b}\right) \to (a, b)$$

namely, the existence of periodic sequences of solutions. Applying two consecutive Laplace transformations in the direction R^1 to the initial solution u of equation (2)

$$\partial_1\partial_2 u = a\partial_1 u + b\partial_2 u$$

and keeping in mind the conditions $A = \frac{1}{a}$, $B = \frac{1}{b}$ (see the example), we obtain a new solution

$$\hat{L}_1(u) = \partial_1^2 u - \left(\frac{\partial_1 b}{b} + b + \frac{1}{b}\right)\partial_1 u + u$$

which does not necessarily coincide with u. In this sense transformation $\hat{L}_1$ is a recursion operator for equation (2). In a similar way one can construct another recursion operator

$$\hat{L}_2(u) = \partial_2^2 u - \left(\frac{\partial_2 a}{a} + a + \frac{1}{a}\right)\partial_2 u + u$$

generated by two consecutive Laplace transformations in the direction R^2. It is an easy exercise to show that in general there are no solutions of period 2, that is, solutions, satisfying any of the equivalent conditions $\hat{L}_1(u) = u$ or $\hat{L}_2(u) = u$. So we will look for solutions of period 4, which obviously can be characterized by the constraint $\hat{L}_1(u) = \hat{L}_2(u)$ or, equivalently,

$$\begin{aligned} \partial_1^2 u &= \left(\frac{\partial_1 b}{b} + b + \frac{1}{b}\right)\partial_1 u + p \\ \partial_2^2 u &= \left(\frac{\partial_2 a}{a} + a + \frac{1}{a}\right)\partial_2 u + p \end{aligned} \tag{2'}$$

for appropriate p. Writing down the compatibility conditions of (2′) with (2) and keeping in mind (14) we obtain the following equations for p:

$$\begin{aligned} \partial_1 p &= \frac{b}{a}\partial_2 u - \partial_1 u + bp \\ \partial_2 p &= \frac{a}{b}\partial_1 u - \partial_2 u + ap. \end{aligned} \tag{2''}$$

Moreover, the compatibility conditions of (2″) are satisfied identically. Hence, any periodic sequence of Laplace equations of period 2 possesses exactly three-dimensional space of periodic solutions of period 4, which are described by an involutive system (2), (2′), (2″). Three linearly independent solutions of this system define a surface M^2 in 3-space, parametrized by coordinates R^1, R^2. This coordinate net is conjugate due to (2) and generates periodic Laplace sequence consisting of four surfaces $M^2 \to M_1^2 \to M_2^2 \to M_3^2 \to M^2$ in the standard differential-geometric sense—see e.g. [17] and references therein for the properties of Laplace sequences of period 4. One can also show, that the second quadratic forms of the surface M^2 and its Laplace images have the special isothermic form in the coordinates R^1, R^2: they are proportional to $(\mathrm{d}R^1)^2 + (\mathrm{d}R^2)^2$, so that all congruences, generating this Laplace sequence, are W-congruences, and the corresponding conjugate nets are the so-called R-nets. Since the radius-vectors of M^2 and M_2^2 satisfy one and the same linear system (2), (2′), (2″), these surfaces differ only by an affine transformation of the 3-space (although do not coincide). The same property holds for M_1^2 and M_3^2. These considerations can be generalized to the case of arbitrary period n as follows. Any periodic equation (2) of an arbitrary period n possesses $(n-1)$-dimensional space of solutions with the same period n—see, e.g. [18].

Let us show that any system of hydrodynamic type, which satisfies equations (15) and generates Laplace sequence of period 2

$$(\lambda^1, \lambda^2) \to (\lambda^2, \lambda^1) \to (\lambda^1, \lambda^2)$$

possesses a unique conservative representation

$$u_t^1 = f_x^1$$
$$u_t^2 = f_x^2$$

of period 4. For that purpose we have to choose the densities u and the fluxes f in such a way that $\hat{L}_1(u) = \hat{L}_2(u)$ and $\hat{L}_1(f) = \hat{L}_2(f)$, where

$$\hat{L}_1(f) = \lambda^1\partial_1^2 u - \left(\frac{\lambda^1\partial_1 b}{b} + \lambda^1 b + \frac{\lambda^2}{b}\right)\partial_1 u + f$$
$$\hat{L}_2(f) = \lambda^2\partial_2^2 u - \left(\frac{\lambda^2\partial_2 a}{b} + \lambda^2 a + \frac{\lambda^1}{a}\right)\partial_2 u + f$$

according to the transformation law of the fluxes, see theorem 1. Finally, we obtain the following system for the conserved densities u

$$\partial_1\partial_2 u = a\partial_1 u + b\partial_2 u$$
$$\partial_1^2 u = \left(\frac{\partial_1 b}{b} + b\right)\partial_1 u - \frac{\partial_2 u}{a}$$
$$\partial_2^2 u = \left(\frac{\partial_2 a}{a} + a\right)\partial_2 u - \frac{\partial_1 u}{b}$$

which can be obtained from (2), (2′), (2″) by the reduction $p = -\frac{\partial_1 u}{b} - \frac{\partial_2 u}{a}$. This system is compatible and defines a 2-parameter space of conserved densities, providing, together with the corresponding fluxes, the unique conservative representation of period 4.

3. Laplace transformations of n-component systems

Lemma. Transformations S_{ij} satisfy the identities

$$\begin{aligned} &S_{ij} \circ S_{ji} = \mathrm{id} \\ &S_{ij} = S_{ik} \circ S_{kj} = S_{kj} \circ S_{ik} \qquad k \neq i, j. \end{aligned} \tag{17}$$

The proof can be obtained by a direct calculation. The analogous identities are well known in the theory of n-conjugate coordinate systems—see, e.g. [5, p 275]. From (17) it immediately follows that transformations S_{ij} form a free Abelian group with $n - 1$ generators (one can take for example $S_{12}, \ldots, S_{1n}$ as the generators).

In the language of n-conjugate coordinate systems transformations S_{ij} have been discussed in [6–10], which partially duplicate the investigations of Darboux [5, pp 274–5]. The summary of these results can be found in [11]. Higher-dimensional Laplace invariants and terminating Laplace sequences have recently been investigated in [13, 14], see also [20, 21]. Let us also point to [12], where the method of factorization has been successfully applied to construct Laplace transformations and Laplace invariants of multidimensional matrix differential operators of the first order.

It looks promising to continue the investigation of transformations S_{ij}, in particular:

• investigate finite families of hydrodynamic-type systems, which are closed under all Laplace transformations S_{ij} (analogues of closed Laplace sequences in the case $n = 2$). It looks likely, that these systems should enjoy the property of certain 'extra' integrability;

• study the behaviour of Hamiltonian structures under the Laplace transformations S_{ij}. We emphasize that local Hamiltonian structures of Dubrovin–Novikov type [2] are not preserved under the Laplace transformations.

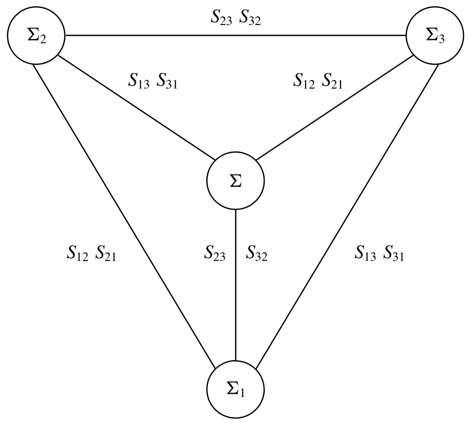

Figure 1.

Let us now give a description of quadruples of 3×3 hydrodynamic-type systems, which are closed under all Laplace transformations. Let the characteristic velocities of the systems Σ, Σ_1, Σ_2, Σ_3 be respectively $(\lambda^1, \lambda^2, \lambda^3)$, $(\lambda^4, \lambda^3, \lambda^2)$, $(\lambda^3, \lambda^4, \lambda^1)$, $(\lambda^2, \lambda^1, \lambda^4)$—see figure 1.

The marked lines joining pairs of systems on the picture indicate, for instance, that system Σ_3 can be obtained from Σ by the Laplace transformations S_{12} and S_{21} (vice versa, system Σ can be obtained from Σ_3 by transformations S_{12} and S_{21}). The lines are marked in accordance with the identities (17). As far as in our construction $S_{ij}^2 = \mathrm{id}$ for any pair of indices i, j, the quadruples of systems under consideration are complete analogues of closed Laplace sequences of period 2. Formulae (8) result in the complicated over-determined system for the characteristic velocities λ^1–λ^4:

$$\begin{aligned}
\frac{\partial_1\partial_2\lambda^1}{\partial_1\lambda^1\partial_2\lambda^1} &= \frac{2}{\lambda^1-\lambda^2} & \qquad \frac{\partial_1\partial_2\lambda^2}{\partial_1\lambda^2\partial_2\lambda^2} &= \frac{2}{\lambda^2-\lambda^1}\\
\frac{\partial_1\partial_3\lambda^1}{\partial_1\lambda^1\partial_3\lambda^1} &= \frac{2}{\lambda^1-\lambda^3} & \qquad \frac{\partial_1\partial_3\lambda^3}{\partial_1\lambda^3\partial_3\lambda^3} &= \frac{2}{\lambda^3-\lambda^1}\\
\frac{\partial_2\partial_3\lambda^2}{\partial_2\lambda^2\partial_3\lambda^2} &= \frac{2}{\lambda^2-\lambda^3} & \qquad \frac{\partial_2\partial_3\lambda^3}{\partial_2\lambda^3\partial_3\lambda^3} &= \frac{2}{\lambda^3-\lambda^2}\\
\frac{\partial_1\partial_2\lambda^3}{\partial_1\lambda^3\partial_2\lambda^3} &= \frac{2}{\lambda^3-\lambda^4} & \qquad \frac{\partial_1\partial_2\lambda^4}{\partial_1\lambda^4\partial_2\lambda^4} &= \frac{2}{\lambda^4-\lambda^3}\\
\frac{\partial_1\partial_3\lambda^2}{\partial_1\lambda^2\partial_3\lambda^2} &= \frac{2}{\lambda^2-\lambda^4} & \qquad \frac{\partial_1\partial_3\lambda^4}{\partial_1\lambda^4\partial_3\lambda^4} &= \frac{2}{\lambda^4-\lambda^2}\\
\frac{\partial_2\partial_3\lambda^1}{\partial_2\lambda^1\partial_3\lambda^1} &= \frac{2}{\lambda^1-\lambda^4} & \qquad \frac{\partial_2\partial_3\lambda^4}{\partial_2\lambda^4\partial_3\lambda^4} &= \frac{2}{\lambda^4-\lambda^1}
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
\partial_1\lambda^2\frac{\lambda^4-\lambda^3}{\lambda^1-\lambda^2} &= \partial_1\lambda^3\frac{\lambda^4-\lambda^2}{\lambda^1-\lambda^3} & \qquad \partial_1\lambda^1\frac{\lambda^2-\lambda^4}{\lambda^3-\lambda^1} &= \partial_1\lambda^4\frac{\lambda^2-\lambda^1}{\lambda^3-\lambda^4}\\
\partial_2\lambda^1\frac{\lambda^4-\lambda^3}{\lambda^2-\lambda^1} &= \partial_2\lambda^3\frac{\lambda^4-\lambda^1}{\lambda^2-\lambda^3} & \qquad \partial_2\lambda^2\frac{\lambda^1-\lambda^4}{\lambda^3-\lambda^2} &= \partial_2\lambda^4\frac{\lambda^1-\lambda^2}{\lambda^3-\lambda^4}\\
\partial_3\lambda^1\frac{\lambda^4-\lambda^2}{\lambda^3-\lambda^1} &= \partial_3\lambda^2\frac{\lambda^4-\lambda^1}{\lambda^3-\lambda^2} & \qquad \partial_3\lambda^3\frac{\lambda^1-\lambda^4}{\lambda^2-\lambda^3} &= \partial_3\lambda^4\frac{\lambda^1-\lambda^3}{\lambda^2-\lambda^4}.
\end{aligned} \tag{19}$$

It follows from (19) that the cross ratio of four characteristic velocities λ^1–λ^4 is constant:

$$\frac{(\lambda^1-\lambda^2)(\lambda^3-\lambda^4)}{(\lambda^1-\lambda^4)(\lambda^3-\lambda^2)}=\mu=\text{constant.}$$

Excluding λ^4 one can rewrite equations (18), (19) in a simplified form

$$\begin{aligned}
&\frac{\partial_1\partial_2\lambda^1}{\partial_1\lambda^1\partial_2\lambda^1}=\frac{2}{\lambda^1-\lambda^2} \qquad &&\frac{\partial_1\partial_2\lambda^2}{\partial_1\lambda^2\partial_2\lambda^2}=\frac{2}{\lambda^2-\lambda^1}\\
&\frac{\partial_1\partial_3\lambda^1}{\partial_1\lambda^1\partial_3\lambda^1}=\frac{2}{\lambda^1-\lambda^3} \qquad &&\frac{\partial_1\partial_3\lambda^3}{\partial_1\lambda^3\partial_3\lambda^3}=\frac{2}{\lambda^3-\lambda^1}\\
&\frac{\partial_2\partial_3\lambda^2}{\partial_2\lambda^2\partial_3\lambda^2}=\frac{2}{\lambda^2-\lambda^3} \qquad &&\frac{\partial_2\partial_3\lambda^3}{\partial_2\lambda^3\partial_3\lambda^3}=\frac{2}{\lambda^3-\lambda^2}
\end{aligned}\tag{20}$$

and

$$\begin{aligned}
&\mu(\lambda^1-\lambda^3)^2\partial_1\lambda^2=(\mu-1)(\lambda^1-\lambda^2)^2\partial_1\lambda^3\\
&\mu(\lambda^2-\lambda^3)^2\partial_2\lambda^1=(\lambda^1-\lambda^2)^2\partial_2\lambda^3\\
&(1-\mu)(\lambda^2-\lambda^3)^2\partial_3\lambda^1=(\lambda^1-\lambda^3)^2\partial_3\lambda^2
\end{aligned}\tag{21}$$

so that (18) and (19) are equivalent to (20) and (21). It follows from (20), that

$$\begin{aligned}
&\partial_1\lambda^1\partial_1\lambda^2=(\lambda^1-\lambda^2)^2\varphi_{13} \qquad &&\partial_2\lambda^1\partial_2\lambda^2=(\lambda^1-\lambda^2)^2\varphi_{23}\\
&\partial_1\lambda^1\partial_1\lambda^3=(\lambda^1-\lambda^3)^2\eta_{12} \qquad &&\partial_3\lambda^1\partial_3\lambda^3=(\lambda^1-\lambda^3)^2\eta_{23}\\
&\partial_2\lambda^2\partial_2\lambda^3=(\lambda^2-\lambda^3)^2 s_{12} \qquad &&\partial_3\lambda^2\partial_3\lambda^3=(\lambda^2-\lambda^3)^2 s_{13}
\end{aligned}\tag{22}$$

where $\varphi_{ij}(R^i,R^j)$, $\eta_{ij}(R^i,R^j)$, $s_{ij}(R^i,R^j)$ are arbitrary functions of the specified arguments. Using (21) one immediately arrives at the following relations between $\varphi_{ij},\eta_{ij},s_{ij}$:

$$\mu\varphi_{13}=(\mu-1)\eta_{12} \qquad \mu\varphi_{23}=s_{12} \qquad (1-\mu)\eta_{23}=s_{13}$$

so that

$$\begin{aligned}
&\varphi_{13}=(\mu-1)\varphi_1(R^1) \qquad \varphi_{23}=\varphi_2(R^2)\\
&\eta_{12}=\mu\varphi_1(R^1) \qquad \eta_{23}=\varphi_3(R^3)\\
&s_{12}=\mu\varphi_2(R^2) \qquad s_{13}=(1-\mu)\varphi_3(R^3)
\end{aligned}$$

where $\varphi_i(R^i)$ are arbitrary functions, which can be reduced to ±1 by the appropriate change of Riemann invariants. In what follows we consider the case $\varphi_i=1$, so that equations (22) assume the form

$$\begin{aligned}
&\partial_1\lambda^1\partial_1\lambda^2=(\mu-1)(\lambda^1-\lambda^2)^2 \qquad &&\partial_2\lambda^1\partial_2\lambda^2=(\lambda^1-\lambda^2)^2\\
&\partial_1\lambda^1\partial_1\lambda^3=\mu(\lambda^1-\lambda^3)^2 \qquad &&\partial_3\lambda^1\partial_3\lambda^3=(\lambda^1-\lambda^3)^2\\
&\partial_2\lambda^2\partial_2\lambda^3=\mu(\lambda^2-\lambda^3)^2 \qquad &&\partial_3\lambda^2\partial_3\lambda^3=(1-\mu)(\lambda^2-\lambda^3)^2.
\end{aligned}\tag{23}$$

We emphasize that (20) and (21) are equivalent to (23). Let us demonstrate that equations (23) are equivalent to a special quadratic reduction of the (2 + 1)-dimensional

3-wave system. For that purpose we introduce differential 1-forms

$$
\begin{aligned}
\omega^1 &= \sqrt{\frac{\mu}{\mu-1}}\frac{\lambda^1-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)}\,\mathrm{d}\lambda^2 - \sqrt{\frac{\mu-1}{\mu}}\frac{\lambda^1-\lambda^2}{(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)}\,\mathrm{d}\lambda^3\\
\omega^2 &= \sqrt{\mu}\frac{\lambda^2-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}\,\mathrm{d}\lambda^1 - \frac{1}{\sqrt{\mu}}\frac{\lambda^1-\lambda^2}{(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)}\,\mathrm{d}\lambda^3\\
\omega^3 &= -\sqrt{\mu-1}\frac{\lambda^2-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}\,\mathrm{d}\lambda^1 - \frac{1}{\sqrt{\mu-1}}\frac{\lambda^1-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)}\,\mathrm{d}\lambda^2
\end{aligned}
\tag{24}
$$

which are chosen in such a way that equations (21) become just

$$
\begin{aligned}
&\omega^1\wedge \mathrm{d}R^2\wedge \mathrm{d}R^3 = 0\\
&\omega^2\wedge \mathrm{d}R^1\wedge \mathrm{d}R^3 = 0\\
&\omega^3\wedge \mathrm{d}R^1\wedge \mathrm{d}R^2 = 0.
\end{aligned}
\tag{25}
$$

Moreover, the forms ω^i satisfy the structure equations of the Lie group $SO(2,1)$:

$$
\mathrm{d}\omega^1 = \omega^2\wedge\omega^3 \qquad \mathrm{d}\omega^2 = \omega^3\wedge\omega^1 \qquad \mathrm{d}\omega^3 = \omega^2\wedge\omega^1. \tag{26}
$$

Let us introduce the coefficients β_{ij} by the formulae

$$
\begin{aligned}
\omega^1 &= \beta_{32}\,\mathrm{d}R^2 - \beta_{23}\,\mathrm{d}R^3\\
\omega^2 &= \beta_{13}\,\mathrm{d}R^3 - \beta_{31}\,\mathrm{d}R^1\\
\omega^3 &= \beta_{12}\,\mathrm{d}R^2 - \beta_{21}\,\mathrm{d}R^1
\end{aligned}
\tag{27}
$$

(the validity of such representation is due to (25)). Coefficients β_{ij} satisfy the nonlinear system, which is well known in the theory of 3-orthogonal coordinates. This system results from the substitution of (27) into the structure equations (26):

$$
\begin{aligned}
&\partial_1\beta_{23} = -\beta_{21}\beta_{13} \qquad \partial_1\beta_{32} = -\beta_{31}\beta_{12}\\
&\partial_2\beta_{13} = -\beta_{12}\beta_{23} \qquad \partial_2\beta_{31} = -\beta_{32}\beta_{21}\\
&\partial_3\beta_{12} = \beta_{13}\beta_{32} \qquad \partial_3\beta_{21} = \beta_{23}\beta_{31}
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
&\partial_1\beta_{12} + \partial_2\beta_{21} + \beta_{31}\beta_{32} = 0\\
&\partial_1\beta_{13} + \partial_3\beta_{31} - \beta_{21}\beta_{23} = 0\\
&\partial_2\beta_{23} + \partial_3\beta_{32} - \beta_{12}\beta_{13} = 0.
\end{aligned}
\tag{29}
$$

The transformation from λ^i to β_{ij} is just the differential substitution of the first order. The explicit expressions for β_{ij} can be obtained by comparing (27) with (24):

$$
\begin{aligned}
\beta_{32} &= \sqrt{\frac{\mu}{\mu-1}}\frac{\lambda^1-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)}\,\partial_2\lambda^2 - \sqrt{\frac{\mu-1}{\mu}}\frac{\lambda^1-\lambda^2}{(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)}\,\partial_2\lambda^3\\
\beta_{23} &= -\sqrt{\frac{\mu}{\mu-1}}\frac{\lambda^1-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)}\,\partial_3\lambda^2 + \sqrt{\frac{\mu-1}{\mu}}\frac{\lambda^1-\lambda^2}{(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)}\,\partial_3\lambda^3\\
\beta_{13} &= \sqrt{\mu}\frac{\lambda^2-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}\,\partial_3\lambda^1 - \frac{1}{\sqrt{\mu}}\frac{\lambda^1-\lambda^2}{(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)}\,\partial_3\lambda^3\\
\beta_{31} &= -\sqrt{\mu}\frac{\lambda^2-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}\,\partial_1\lambda^1 + \frac{1}{\sqrt{\mu}}\frac{\lambda^1-\lambda^2}{(\lambda^1-\lambda^3)(\lambda^2-\lambda^3)}\,\partial_1\lambda^3
\end{aligned}
$$

$$\beta_{12} = -\sqrt{\mu-1}\frac{\lambda^2-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}\,\partial_2\lambda^1 - \frac{1}{\sqrt{\mu-1}}\frac{\lambda^1-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)}\,\partial_2\lambda^2$$
$$\beta_{21} = \sqrt{\mu-1}\frac{\lambda^2-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^1-\lambda^3)}\,\partial_1\lambda^1 + \frac{1}{\sqrt{\mu-1}}\frac{\lambda^1-\lambda^3}{(\lambda^1-\lambda^2)(\lambda^2-\lambda^3)}\,\partial_1\lambda^2.$$

Moreover, equations (23) impose the following quadratic reduction on the coefficients β_{ij}:

$$\frac{\beta_{32}^2}{\mu} - \beta_{12}^2 = -4 \qquad \frac{\beta_{23}^2}{\mu-1} - \beta_{13}^2 = 4 \qquad \frac{\beta_{21}^2}{\mu-1} - \frac{\beta_{31}^2}{\mu} = 4. \tag{30}$$

Let us point out that equations (28) are just $(2+1)$-dimensional 3-wave systems, corresponding to the spectral problem

$$\begin{aligned} &\partial_2 H_1 = -\beta_{21}H_2 \qquad \partial_3 H_1 = -\beta_{31}H_3 \\ &\partial_1 H_2 = -\beta_{12}H_1 \qquad \partial_3 H_2 = -\beta_{32}H_3 \\ &\partial_1 H_3 = \beta_{13}H_1 \qquad \partial_2 H_3 = \beta_{23}H_2. \end{aligned}$$

In order to comply with reduction (30) we introduce the parametrization

$$\begin{aligned} &\beta_{32} = 2\sqrt{\mu}\,\mathrm{sh}\,u \qquad \beta_{12} = 2\,\mathrm{ch}\,u \\ &\beta_{23} = 2\sqrt{\mu-1}\,\mathrm{ch}\,v \qquad \beta_{13} = 2\,\mathrm{sh}\,v \\ &\beta_{21} = 2\sqrt{\mu-1}\,\mathrm{ch}\,w \qquad \beta_{31} = 2\sqrt{\mu}\,\mathrm{sh}\,w \end{aligned}$$

so that equations (28) become

$$\begin{aligned} &\partial_1 u = -2\,\mathrm{sh}\,w \qquad \partial_3 u = 2\sqrt{\mu}\,\mathrm{sh}\,v \\ &\partial_1 v = -2\,\mathrm{ch}\,w \qquad \partial_2 v = -2\sqrt{\mu-1}\,\mathrm{ch}\,u \\ &\partial_2 w = -2\sqrt{\mu-1}\,\mathrm{sh}\,u \qquad \partial_3 w = 2\sqrt{\mu}\,\mathrm{ch}\,v \end{aligned} \tag{31}$$

(we point out that after this substitution equations (29) are satisfied identically). After the appropriate rescaling of Riemann invariants equations (31) assume the simple form

$$\begin{aligned} &\partial_1 u = \mathrm{sh}\,w \qquad \partial_3 u = \mathrm{sh}\,v \\ &\partial_1 v = \mathrm{ch}\,w \qquad \partial_2 v = \mathrm{ch}\,u \\ &\partial_2 w = \mathrm{sh}\,u \qquad \partial_3 w = \mathrm{ch}\,v. \end{aligned}$$

Expressing v and w as follows

$$v = \mathrm{arcsh}\,\partial_3 u \qquad w = \mathrm{arcsh}\,\partial_1 u$$

one can rewrite this system in the form of three pairwise commuting Monge–Ampere equations

$$\begin{aligned} &\partial_1\partial_2 u = \mathrm{sh}\,u\sqrt{1+(\partial_1 u)^2} \\ &\partial_3\partial_2 u = \mathrm{ch}\,u\sqrt{1+(\partial_3 u)^2} \\ &\partial_1\partial_3 u = \sqrt{1+(\partial_1 u)^2}\sqrt{1+(\partial_3 u)^2} \end{aligned}$$

(as far as the authors know the problem of classification of commuting Monge–Ampere equations has not been addressed before). It looks promising to continue the investigation of finite families of $n \times n$ hydrodynamic-type systems in Riemann invariants, which are closed under all Laplace transformations S_{ij}. It is natural to restrict oneself to the case when $S_{ij}^k = \mathrm{id}$ for some $k \geqslant 2$. Examples discussed above support the evidence that the problem is nontrivial even in the simplest cases $(n = 2, k = 2)$ and $(n = 3, k = 2)$. The equations for the characteristic velocities of the corresponding hydrodynamic-type systems should reduce to appropriate integrable reductions of the $(2+1)$-dimensional n-wave system

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj}.$$

4. Laplace transformations of the Lame and rotation coefficients

In the 2-component case the Lame coefficients h_1, h_2 are defined by the formulae

$$\partial_2 \ln h_1 = a \qquad \partial_1 \ln h_2 = b. \tag{32}$$

It turns out that Laplace transformation (3) can be pulled back to the transformation of the Lame coefficients: the transformed Lame coefficients H_1, H_2 are given by the formulae

$$\begin{aligned} H_1 &= \frac{h_1}{b} = \frac{h_1 h_2}{\partial_1 h_2} \\ H_2 &= h_2 A = h_2 \partial_2 \ln H_1 \end{aligned} \tag{33}$$

(see Darboux [5]), so that

$$\partial_2 \ln H_1 = A \qquad \partial_1 \ln H_2 = B.$$

In terms of the chain

$$\begin{aligned} \partial_1 \ln h_2^n &= \frac{h_1^n}{h_1^{n+1}} \\ \partial_2 \ln h_1^n &= \frac{h_2^n}{h_2^{n-1}} \end{aligned} \tag{34}$$

transformation (33) reduces to the shift

$$(h_1^n, h_2^n) \to (h_1^{n+1}, h_2^{n+1}).$$

Formulae (33) can also be rewritten in terms of the rotation coefficients

$$\beta_{12} = \frac{\partial_1 h_2}{h_1} \qquad \beta_{21} = \frac{\partial_2 h_1}{h_2} \tag{35}$$

namely, the transformed rotation coefficients $\tilde{\beta}_{12}, \tilde{\beta}_{21}$ are the following:

$$\begin{aligned} \tilde{\beta}_{12} &= \frac{\partial_1 H_2}{H_1} = \beta_{12}(\beta_{12}\beta_{21} - \partial_1 \partial_2 \ln \beta_{12}) \\ \tilde{\beta}_{21} &= \frac{\partial_2 H_1}{H_2} = \frac{1}{\beta_{12}}. \end{aligned} \tag{36}$$

The generalization to the n-component case is straightforward: under the transformation S_{ij} the Lame coefficients h_i defined by the formula

$$\partial_j \ln h_i = a_{ij}$$

transform into H_i as follows

$$\begin{aligned} H_i &= \frac{h_i}{a_{ji}} = \frac{h_i h_j}{\partial_i h_j} \\ H_j &= h_j A_{ij} = h_j \partial_j \ln H_i \\ H_k &= h_k \left(1 - \frac{a_{ki}}{a_{ji}}\right) = h_k - \frac{\partial_i h_k}{\partial_i h_j} h_j \qquad k \neq i, j. \end{aligned} \tag{37}$$

Indeed one can directly verify that

$$\partial_j \ln H_i = A_{ij}$$

where A_{ij} are specified by (10).

In a similar way, the rotation coefficients

$$\beta_{ij} = \frac{\partial_i h_j}{h_i}$$

transform into $\tilde{\beta}_{ij}$ as follows

$$\begin{aligned}
\tilde{\beta}_{ij} &= \beta_{ij}(\beta_{ij}\beta_{ji} - \partial_i\partial_j \ln \beta_{ij}) \\
\tilde{\beta}_{ji} &= \frac{1}{\beta_{ij}} \\
\tilde{\beta}_{ik} &= -\beta_{ij}\partial_i \frac{\beta_{ik}}{\beta_{ij}} \\
\tilde{\beta}_{ki} &= \frac{\beta_{kj}}{\beta_{ij}} \\
\tilde{\beta}_{jk} &= -\frac{\beta_{ik}}{\beta_{ij}} \\
\tilde{\beta}_{kj} &= \beta_{ij}\partial_j \frac{\beta_{kj}}{\beta_{ij}} \\
\tilde{\beta}_{kl} &= \beta_{kl} - \frac{\beta_{kj}\beta_{il}}{\beta_{ij}}
\end{aligned} \tag{38}$$

where $k, l \neq i, j$ (compare with [19, p 12]). In order to check that indeed

$$\tilde{\beta}_{ij} = \frac{\partial_i H_j}{H_i}$$

it is convenient to use the following equivalent representation for the transformed rotation coefficients H_i:

$$\begin{aligned}
H_i &= \frac{h_j}{\beta_{ij}} \\
H_j &= \partial_j h_j - h_j \partial_j \ln \beta_{ij} \\
H_k &= h_k - \frac{\beta_{ik}}{\beta_{ij}} h_j \qquad k \neq i, j
\end{aligned}$$

and to keep in mind the equations

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj} \qquad i \neq j \neq k \tag{39}$$

satisfied by the rotation coefficients of an arbitrary semi-Hamiltonian system. It should be emphasized that the transformed rotation coefficients satisfy the same equations (39) and hence transformations S_{ij} written in the form (38) are just discrete symmetries of the $(2+1)$-dimensional n-wave system (39). The role of these transformations in the theory of n-wave system deserves a special investigation. Basically transformations S_{ij} preserve neither the Egorov reduction

$$\beta_{ij} = \beta_{ji}$$

nor the zero curvature reduction

$$\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{k \neq i, j} \beta_{ki}\beta_{kj} = 0$$

of the n-wave system (39).

Remark. A way to generalize transformations S_{ij} written in the form (37), (38) is to allow the rescaling

$$H_s \to \mu_s H_s \qquad \tilde{\beta}_{sl} \to \frac{\mu_l}{\mu_s}\tilde{\beta}_{sl} \qquad \mu_s = \text{constant}$$

where generically the scaling factors μ_s depend on S_{ij}. One can always choose μ_s in such a way as to preserve the basic identities (17). The main purpose for introducing the scaling factors is the construction of solutions of the n-wave system (39) for which, say, $S_{ij}^2 = \text{id}$ (analogues of periodic sequences of period 2). For instance, one can show that when $n \geqslant 3$ there are no nontrivial solutions of (39) satisfying $S_{ij}^2 = \text{id}$ if S_{ij} are as in (38). However, they do exist if we rescale S_{ij} appropriately.

5. Laplace transformations as symmetries of the Davey–Stewartson-type equations

It should be noted that Laplace transformation (5) for the characteristic velocities

$$\Lambda^1 = \lambda^2$$
$$\Lambda^2 = \lambda^2 - \frac{1}{\frac{\partial_1\partial_2\lambda^2}{\partial_1\lambda^2\partial_2\lambda^2} + \frac{1}{\lambda^1-\lambda^2}}$$

arises naturally as the symmetry of the $(2+1)$-dimensional integrable systems

$$r_t = r_{xx} - \frac{2r_x^2}{r-s} + 2r_x R$$
$$s_t = -s_{xx} - \frac{2s_x^2}{r-s} + 2s_x R$$
$$R_y = \frac{r_x s_y - r_y s_x}{(r-s)^2}$$

and

$$r_t = r_{yy} - \frac{2r_y^2}{r-s} + 2r_y S$$
$$s_t = -s_{yy} - \frac{2s_y^2}{r-s} + 2s_y S$$
$$S_x = \frac{r_y s_x - r_x s_y}{(r-s)^2}$$

which have recently been discussed in [4] and are both equivalent to the Davey–Stewartson equations. These equations are invariant under the transformation

$$\tilde{r} = s \qquad \tilde{s} = s - \frac{1}{\frac{s_{xy}}{s_x s_y} + \frac{1}{r-s}}$$

if we define

$$\tilde{R} = R + \left(\ln\frac{s_y}{s_x}\right)_x \qquad \tilde{S} = S + \left(\ln\frac{s_x}{s_y}\right)_y .$$

Formulae for $\tilde{r}, \tilde{s}$ coincide with (5) after introducing the notation

$$r = \lambda^1 \qquad s = \lambda^2 \qquad \partial_1 = \partial_x \qquad \partial_2 = \partial_y .$$

In a similar way Laplace transformation (36) for the rotation coefficients

$$\tilde\beta_{12} = \beta_{12}(\beta_{12}\beta_{21} - \partial_1\partial_2 \ln\beta_{12})$$
$$\tilde\beta_{21} = \frac{1}{\beta_{12}}$$

arises as the symmetry of the Davey–Stewartson equations

$$\mathrm{i}u_t = u_{xx} + u_{yy} - u(p+q)$$
$$-\mathrm{i}v_t = v_{xx} + v_{yy} - v(p+q)$$
$$p_x = 2(uv)_y \qquad q_y = 2\partial_x(uv)$$

which were shown in [15] to be invariant under the transformation

$$\tilde u = u(uv - (\ln u)_{xy})$$
$$\tilde v = \frac{1}{u}$$
$$\tilde p = p - 2(\ln u)_{yy}$$
$$\tilde q = q - 2(\ln u)_{xx}.$$

Formulae for $\tilde u, \tilde v$ coincide with (36) after introducing the notation

$$\beta_{12} = u \qquad \beta_{21} = v \qquad \partial_1 = \partial_x \qquad \partial_2 = \partial_y.$$

6. Geometric background

In this section we give a geometric interpretation of formulae (5) based on the correspondence between systems of conservation laws and line congruences in the projective space.

Let us consider a 2×2 system (1) in Riemann invariants

$$R^1_t = \lambda^1(R)R^1_x$$
$$R^2_t = \lambda^2(R)R^2_x$$

and choose its conservative representation

$$u^1_t = f^1_x \qquad u^2_t = f^2_x \tag{40}$$

where $u = (u^1(R), u^2(R))$ are conserved densities of system (1) with the corresponding fluxes $f = (f^1(R), f^2(R))$. We recall that the densities and the fluxes satisfy the equations

$$\partial_i f = \lambda^i \partial_i u \qquad i = 1, 2.$$

Following [16], we associate with (40) a congruence of straight lines in the 3-space $E^3(y^0, y^1, y^3)$ defined by the formulae

$$y^1 = u^1y^0 - f^1 \qquad y^2 = u^2y^0 - f^2. \tag{41}$$

This correspondence was investigated in [16], where it was shown that all familiar constructions in the theory of systems of conservation laws (40) have their natural geometric counterpart in projective theory of congruences. Let us recall the definition of the Laplace transformation of congruence (41). Any congruence (41) has two focal surfaces with the radius-vectors $\boldsymbol{r}$ and $\boldsymbol{R}$:

$$\boldsymbol{r} = \begin{pmatrix} \lambda^1 \\ \lambda^1 u^1 - f^1 \\ \lambda^1 u^2 - f^2 \end{pmatrix} \qquad \boldsymbol{R} = \begin{pmatrix} \lambda^2 \\ \lambda^2 u^1 - f^1 \\ \lambda^2 u^2 - f^2 \end{pmatrix}.$$

The curves $R^2 = \text{constant}$ and $R^1 = \text{constant}$ are conjugate on both of the focal surfaces. The lines of our congruence (41) are tangent to the curves $R^2 = \text{constant}$ on the focal surface with the radius-vector $\boldsymbol{r}$ and can be represented parametrically as follows

$$\boldsymbol{y} = \boldsymbol{r} + t\partial_1 \boldsymbol{r}$$

so that equations (41) can be obtained by excluding parameter t. The Laplace transformation (in the direction R^1) of congruence (41) is a congruence, formed by the tangents to the curves $R^2 = \text{constant}$ on the second focal surface $\boldsymbol{R}$:

$$\boldsymbol{y} = \boldsymbol{R} + t\partial_1 \boldsymbol{R}$$

or, in the components,

$$\begin{aligned} y^0 &= \lambda^2 + t\partial_1\lambda^2 \\ y^1 &= \lambda^2 u^1 - f^1 + t(\partial_1\lambda^2 u^1 + (\lambda^2 - \lambda^1)\partial_1 u^1) \\ y^2 &= \lambda^2 u^2 - f^2 + t(\partial_1\lambda^2 u^2 + (\lambda^2 - \lambda^1)\partial_1 u^2). \end{aligned}$$

Excluding t, we can rewrite these equations in the form

$$\begin{aligned} y^1 &= U^1 y^0 - F^1 \\ y^2 &= U^2 y^0 - F^2 \end{aligned}$$

where

$$\begin{aligned} U^1 &= u^1 + \frac{(\lambda^2 - \lambda^1)\partial_1 u^1}{\partial_1\lambda^2} = u^1 - \frac{\partial_1 u^1}{b} \\ U^2 &= u^2 + \frac{(\lambda^2 - \lambda^1)\partial_1 u^2}{\partial_1\lambda^2} = u^2 - \frac{\partial_1 u^2}{b} \\ F^1 &= f^1 + \frac{\lambda^2(\lambda^2 - \lambda^1)\partial_1 u^1}{\partial_1\lambda^2} = f^1 - \frac{\lambda^2\partial_1 u^1}{b} \\ F^2 &= f^2 + \frac{\lambda^2(\lambda^2 - \lambda^1)\partial_1 u^2}{\partial_1\lambda^2} = f^2 - \frac{\lambda^2\partial_1 u^2}{b} \end{aligned}$$

(we recall that $a = \frac{\partial_2\lambda^1}{\lambda^2 - \lambda^1}$, $b = \frac{\partial_1\lambda^2}{\lambda^1 - \lambda^2}$). The system of conservation laws

$$\begin{aligned} U^1_t &= F^1_x \\ U^2_t &= F^2_x \end{aligned} \tag{42}$$

is called the Laplace transform of system (40). In Riemann invariants the transformed system assumes the form

$$\begin{aligned} R^1_t &= \Lambda^1(R) R^1_x \\ R^2_t &= \Lambda^2(R) R^2_x \end{aligned}$$

(we point out that the transformed system (42) has the same Riemann invariants as (40)), where the new characteristic velocities are given by formula (5):

$$\begin{aligned} \Lambda^1 &= \lambda^2 \\ \Lambda^2 &= \lambda^2 - \frac{b\partial_2\lambda^2}{\partial_2 b - ab} = \lambda^2 - \frac{1}{\frac{\partial_1\partial_2\lambda^2}{\partial_1\lambda^2\partial_2\lambda^2} + \frac{1}{\lambda^1 - \lambda^2}}. \end{aligned}$$

We emphasize, that the Laplace transformation of the characteristic velocities does not depend on the particular conservative representation (40) of the given system (1).

Acknowledgments

I would like to thank the participants of the seminar 'Geometry and Mathematical Physics' in the Moscow State University for fruitful discussions and N Kamran for sending me the reprints of [13, 14]. I also take the opportunity to thank Y Nutku for the invitation to the TÜBITAK-Marmara Research Center where part of this work was completed.

This research was partially supported by RBFR-DFG grant N 96-01-00050 and the Aleksander von Humboldt Foundation.

References

[1] Tsarev S P 1990 *Izv AN SSSR* **54** 1048–68
[2] Dubrovin B A and Novikov S P 1983 *Dokl. AN SSSR* **27** 665
[3] Veselov A P and Novikov S P 1995 *Usp. Mat. Nauk.* **6** 171–2
[4] Shabat A B and Yamilov R I 1997 *Phys. Lett.* A **227** 15–23
[5] Darboux G 1914 *Lecons sur la Theorie Generale des Surfaces* part 4 (Paris: Gauthier-Villars)
[6] Chern S S 1944 *Proc. Natl Acad. Sci.* **30** 95–7
[7] Koz'mina T L 1947 *Dokl. AN SSSR* **55** 187–9
[8] Smirnov R V 1950 *Dokl. AN SSSR* **71** 437–9
[9] Shul'man T A 1952 *Dokl. AN SSSR* **85** 501–4
[10] Akivis M A and Goldberg V V 1991 *Rend. Sem. Mat. Messina, Ser. II* **1** 9–29
[11] Akivis M A and Goldberg V V 1993 *Projective Differential Geometry of Submanifolds (Math. Library 49)* (Amsterdam: North-Holland)
[12] Athorne C 1995 *Phys. Lett.* A **206** 162–6
[13] Kamran N and Tenenblat K 1996 *Duke Math. J.* **84** 237–66
[14] Kamran N and Tenenblat K 1997 Hydrodynamic systems and the higher-dimensional Laplace transformation of Cartan submanifolds *Proc. Conf. 'Algebraic methods and theoretical Physics' (CRM Conf. Proc. Series)* ed L Vinet and Y Saint-Aubin (Providence, RI: American Mathematical Society)
[15] Leznov A N, Shabat A B and Yamilov R I 1993 *Phys. Lett.* A **174** 397–402
[16] Agafonov S I and Ferapontov E V 1996 *Izv. AN SSSR*
[17] Bol G 1967 *Projective Differentialgeometrie* vol 3
[18] Tzitzeica G 1924 *Geometrie Differentielle Projective des Reseaux* (Paris: Academie Roumaine)
[19] Guichard C 1905 *Sur les Systèmes Triplement Indéterminés et sur les Systèmes Triple-orthogonaux* (Paris: Collection Scientia)
[20] Eisenhart L P 1919 *Ann. Math.* **20** 262–73
[21] Korovin V I 1955 *Dokl. AN SSSR* **101** 797–9